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On integrable deformations of the spherical top

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Abstract. The motion on the sphere S^2 in potential $V = (x_1x_2x_3)^{-2/3}$ is considered. The Lax representation and the linearization procedure for this two-dimensional integrable system are discussed.

1. Description of the model

The system under consideration is a special case of the following mechanical system in the nine-dimensional space $\mathbb{R}^9$:

$$F^*(t)\frac{\mathrm{d}^2F(t)}{\mathrm{d}t^2} + |\det F(t)|^{1-\gamma}G = 0. \tag{1}$$

Here $F(t)$ and G are 3×3 matrices, F^* denotes the transpose matrix and γ is a polytropic index. The F_{jk} components of the matrix F are coordinates on the configuration space $\mathbb{R}^9$. These equations of motion have been studied many authors, see [2, 6, 8, 12].

According to [12], we shall only consider symmetric constant matrices $G = G^*$. In this case, by using the canonical transformation of variables $F' = UFU^*$, we can reduce the constant matrix G to the diagonal matrix with the following diagonal elements: ± 1 or 0. Moreover, from a physical point of view we can put $G = I$ without loss of generality [12].

The Newton equations (1) arise in the solution of the hydrodynamical equations representing the dynamics of a cloud of compressible gas expanding freely in an otherwise empty space. This model has a rich history associated with well known researchers such as Dirichlet, Dedekind and Riemann. For an extensive discussion of the model we refer the reader to [6], a book which should be viewed as a general reference guide to the subject.

At $G = G^*$ the invariance of the problem under the rotation and internal motion of the gas leads to conservation of the angular momentum and the vorticity operator:

$$\boldsymbol{J} = F(t)\dot F^*(t) - \dot F(t)F^*(t) \qquad \boldsymbol{K} = F^*(t)\dot F(t) - \dot F^*(t)F(t).$$

Both $\boldsymbol{J}$ and $\boldsymbol{K}$ are antisymmetric matrices with three independent components. Thus, equations (1) possess an enlarged symmetry group $SO(4) \simeq SO(3)\times SO(3)$. There is also a discrete symmetry, which allows the vorticity and the angular momentum to be interchanged. This discrete symmetry is identical to the duality principle of Dedekind [8].

For the perfect monatomic gas, at $\gamma = \frac{5}{3}$ the system of equations (1) possesses one more integral of motion [2]:

$$\boldsymbol{r}^2 = \mathrm{tr}(F^*(t)F(t)) \qquad \text{at} \quad \gamma = \tfrac{5}{3}. \tag{2}$$

The qualitative behaviour of solutions at the different values of the adiabatic index γ and some partial cases of motion are discussed in [6].

It is easy to separate the diagonal and non-diagonal components of equations (1) [6,8,12]. The non-diagonal components give six kinematical equations only involving the inertial properties of the gas cloud and not the pressure force. The diagonal components give three dynamical equations determining the rate of expansion of the gas cloud under the influence of the pressure force.

Below, at $\gamma = \frac{5}{3}$, we consider a free expansion of an ellipsoidal gas cloud with fixed orientation, having zero angular momentum and zero vorticity $\boldsymbol{K} = \boldsymbol{J} = 0$. In this case matrix $F(t)$ is diagonal for all t:

$$F(t) = \mathrm{diag}(x_1, x_2, x_3)(t)$$

and the constant matrix $G = I$ is equal to unity. The corresponding three equations of motion are given by

$$x_1\ddot{x}_1 = x_2\ddot{x}_2 = x_3\ddot{x}_3 = \frac{\mathrm{const}}{(x_1x_2x_3)^{2/3}} \qquad \text{at} \quad \gamma = \tfrac{5}{3}. \tag{3}$$

The additional integral of motion (2) [2] is equal to the radius of the sphere $\boldsymbol{r}^2 = \sum x_k^2$. By using this integral our system on $\mathbb{R}^3$ may be reduced to the system on the sphere S^2. At $\gamma = \frac{5}{3}$ the third independent integral of motion was derivd in [9].

The reduced system has a configuration space diffeomorphic to the Euclidean motion group $E(3) = SO(3) \times \mathbb{R}^3$ [3]. It allows one to identify the phase space of this system on T^*S^2 with the cotangent bundle $T^*E(3)$. The kinetic energy is a left-invariant Riemannian metric on $E(3)$. It is determined by some quadratic form on the dual space $e^*(3)$ of the Lie algebra $e(3)$ [3, 13].

By using the Killing form the dual space $e^*(3)$ may be identified with algebra $e(3) = so(3) \oplus \mathbb{R}^3$, the semi-direct sum of $so(3)$ and the Abelian space $\mathbb{R}^3$. Let two vectors $J \in so(3) \simeq \mathbb{R}^3$ and $x \in \mathbb{R}^3$ be coordinates in the dual space $e^*(3)$ equipped with natural Lie–Poisson brackets

$$\begin{aligned} &\{J_i, J_j\} = \varepsilon_{ijk}J_k \qquad \{J_i, x_j\} = \varepsilon_{ijk}x_k \\ &\{x_i, x_j\} = 0 \qquad i, j, k = 1, 2, 3. \end{aligned} \tag{4}$$

Here ε_{ijk} is the standard totally skew-symmetric tensor. The generic coadjoint orbits of $E(3)$ in $e^*(3)$ are four-dimensional symplectic leaves specified by the two Casimir elements

$$\boldsymbol{C}_1 = (x, x) = x_ix_i \qquad \boldsymbol{C}_2 = (J, x) = J_ix_i. \tag{5}$$

Here (x, y) means the inner product in $\mathbb{R}^3$. Thus, the dual space $e^*(3)$ decomposes into the coadjoint orbits

$$\mathcal{O}_{c_1,c_2} = \{\{J, x\} \in \mathbb{R}^6 : \boldsymbol{C}_1 = c_1, \boldsymbol{C}_2 = c_2\} \tag{6}$$

which are invariant with respect to the usual Euler–Poisson equations in $e^*(3)$ [3, 13].

Let us introduce a complex analogue of the Lie algebra $e(3)$, as a semi-direct sum of $so(3, \mathbb{C})$ and the complex space $\mathbb{C}^3$. This algebra $e(3, \mathbb{C}) = so(3, \mathbb{C}) \oplus \mathbb{C}^3$ is equipped with the same Lie–Poisson brackets (4) and Casimir operators (5). In contrast to the usual $e(3)$ algebra, it allows us to consider non-trivial representations at the zero value $c_1 = 0$ of the first Casimir operator $\boldsymbol{C}_1$.

The condition $c_1 = 0$ has no obvious physical or geometric meaning. Of course, we cannot consider a real sphere of zero radius, but from a mathematical point of view c_1 is an arbitrary value of the Casimir element. Note, that 'non-physical' representations of the algebra $sl(2)$ with the zero-spin $s = 0$ are also useful in physics [11, 15].

Proposition 1. *At the zero value $c_1 = 0$ of the first Casimir operator $\boldsymbol{C}_1$ (5) the following transformation in $so^*(3, \mathbb{C}) \subset e^*(3, \mathbb{C})$:*

$$J \to \tilde{J} = J + \frac{\mathrm{i}a}{(x_1 x_2 x_3)^{1/3}} x \qquad a \in \mathbb{C} \tag{7}$$

is an outer automorphism of the representation of $e(3, \mathbb{C})$.

By using embedding $e(3) \subset e(3, \mathbb{C})$ let us consider known integrable tops on this complex algebra $e^*(3, \mathbb{C})$. Applying transformation (7) one can get integrable deformations of these tops on the one-parameter set of orbits $\mathcal{O}_1$ ($c_1 = 0, c_2 = \text{const}$). Sometimes outer automorphism (7) allows us to get much more.

As an example, let us consider a spherical top with the standard Hamilton function

$$H = (\tilde{J}, \tilde{J}) = \tilde{J}_1^2 + \tilde{J}_2^2 + \tilde{J}_3^2 \tag{8}$$

and with the non-standard second integral of motion

$$K = \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 \tag{9}$$

defined only on the subalgebra $so(3)$. Of course, by substituting vector $\tilde{J}$ (7) one gets an integrable deformation of this symmetric top at $c_1 = 0$. However, we can prove the following proposition.

Proposition 2. *Outer automorphism (7) maps Hamiltonian (8) of the spherical top into the following Hamiltonian:*

$$H = \sum_{k=1}^{3} J_k^2 + 2\mathrm{i}a \frac{c_2}{(x_1 x_2 x_3)^{1/3}} - a^2 \frac{c_1}{(x_1 x_2 x_3)^{2/3}}. \tag{10}$$

The proposed deformation of the spherical top is completely integrable on both the one-parameter sets of orbits

$$\mathcal{O}_1 : (c_1 = 0, c_2 = \text{const}) \qquad \textit{and} \qquad \mathcal{O}_2 : (c_1 = \text{const}, c_2 = 0).$$

Moreover, to consider the second orbits we can return to the usual real Lie algebra $e^*(3)$.

At $c_1 = 0$ the second integral is the image of known integral K (9)

$$\begin{aligned} K = J_1 J_2 J_3 - a^2 \left(\frac{J_1}{x_1} + \frac{J_2}{x_2} + \frac{J_3}{x_3} \right) (x_1 x_2 x_3)^{1/3} \\ + \frac{2\mathrm{i}a}{(x_1 x_2 x_3)^{1/3}} (J_1 J_2 x_3 + J_1 x_2 J_3 + x_1 J_2 J_3) - \mathrm{i}a^3. \end{aligned} \tag{11}$$

At $c_2 = 0$ the Hamilton function (10) is in involution with the following second integral of motion:

$$K = J_1 J_2 J_3 + a^2 \left(\frac{J_1}{x_1} + \frac{J_2}{x_2} + \frac{J_3}{x_3} \right) (x_1 x_2 x_3)^{1/3}. \tag{12}$$

In contrast with (11), here we removed the imaginary terms and changed the sign before the rest potential term. We do not know the origin of such an additional transformation at present.

In the natural variables

$$y = \frac{x}{(x_1 x_2 x_3)^{1/3}} \qquad y_1 y_2 y_3 = 1$$

transformation (7) becomes a shift $\tilde{J} = J + \mathrm{i}ay$ and the integrals of motion (10) and (12) are given by

$$\begin{aligned} H &= J_1^2 + J_2^2 + J_3^2 - a^2 (y_1^2 + y_2^2 + y_3^2) \\ K &= J_1 J_2 J_3 + a^2 \left(\frac{J_1}{y_1} + \frac{J_2}{y_2} + \frac{J_3}{y_3} \right). \end{aligned} \tag{13}$$

The Euler–Poisson equations on $e^*(3)$ generated by (10) are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}J &= \frac{2a^2}{3}(y,y)y\times y^{-1} \qquad & y^{-1} &= (y_1^{-1}, y_2^{-1}, y_3^{-1}) \\ \frac{\mathrm{d}}{\mathrm{d}t}y &= -\frac{2}{3}(y,y)J\times y^{-1} \qquad & (y,y) &= y_1^2+y_2^2+y_3^2 \end{aligned} \tag{14}$$

where $x \times y$ means the standard vector product in $\mathbb{R}^3$. Thus, we rewrite the initial very symmetric equations of motion (3) defined on the configuration space $\mathbb{R}^3$ as the Euler–Poisson equations (14) defined on the phase space $e^*(3)$.

2. Lax representation

The main purpose of this paper is to rewrite the equations of motion (14) in the Lax form

$$\frac{\mathrm{d}}{\mathrm{d}t}L = [L, M]. \tag{15}$$

Let us briefly recall the construction of the Lax pair for the Neumann system. The Neumann system is an integrable system on the sphere with quadratic potential (see (13)). Its phase space may be modelled on the dual space $e^*(3)$ at $c_2 = 0$. The corresponding Euler–Poisson equations are equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}J = x\times z \qquad \frac{\mathrm{d}}{\mathrm{d}t}x = -J\times x \qquad z = -\,\mathrm{diag}(a_1,a_2,a_3)x \tag{16}$$

where a_j are arbitrary parameters.

The Neumann system possesses the necessary number of quadratic integrals of motion. Nevertheless, the Lax pair cannot be constructed in the framework of the algebra $e(3) = so(3) \oplus \mathbb{R}^3$. Namely, for the Neumann system and some others system, we have to use the Cartan-type decomposition of the Lie algebra $gl(3,\mathbb{R}) = so(3) + Symm(3)$ [13].

Let us introduce the antisymmetric matrix of angular momentum $\mathcal{J} \in so(3)$ and the symmetric matrix of coordinates $\mathcal{X} \in Symm(3)$

$$\begin{aligned} &\mathcal{J} \in so(3) \simeq \mathbb{R}^3: \qquad && \mathcal{J}_{ij} = \varepsilon_{ijk}\mathcal{J}_k \\ &\mathcal{X} \in Symm(3): \qquad && \mathcal{X}_{ij} = x_i x_j. \end{aligned}$$

Then the Lax representations for the Neumann system are given by

$$L = \mathrm{diag}(a_1,a_2,a_3)\lambda + \mathcal{J} + \lambda^{-1}\mathcal{X} \qquad M = -\lambda^{-1}\mathcal{X}. \tag{17}$$

Let us present these Lax matrices explicitly:

$$L = \begin{pmatrix} a_1\lambda & 0 & 0 \\ 0 & a_2\lambda & 0 \\ 0 & 0 & a_3\lambda \end{pmatrix} + \begin{pmatrix} 0 & -J_3 & J_2 \\ J_3 & 0 & -J_1 \\ -J_2 & J_1 & 0 \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{pmatrix}$$

$$M = -\frac{x_1x_2x_3}{\lambda}\begin{pmatrix} \frac{x_1}{x_2x_3} & x_3^{-1} & x_2^{-1} \\ x_3^{-1} & \frac{x_2}{x_1x_3} & x_1^{-1} \\ x_2^{-1} & x_1^{-1} & \frac{x_3}{x_1x_2} \end{pmatrix}.$$

Now let us turn to the deformation of the completely symmetric top (10). The Lie algebras $\mathbb{R}^3$ with the vector product, and $so(3)$ with the usual commutator may be identified by using the Lie algebras isomorphism

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \longrightarrow \mathcal{X} = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \in so(3).$$

In (7) the element $J \in so(3)$ has been added with the vector $x \in \mathbb{R}^3$. Thus, defining the outer automorphism (7), we implicitly used this property of the three-dimensional Euclidean space. Below we use the same property to construct the Lax representation.

The main recipe is to rearrange items in the decomposition $gl(3,\mathbb{R}) = so(3) + Symm(3)$. Let us introduce a symmetric matrix of angular momentum $\mathcal{J} \in Symm(3)$ and an antisymmetric matrix of coordinates $\mathcal{X} \in so(3)$

$$\mathcal{J} \in Symm(3): \qquad \mathcal{J}_{ij} = |\varepsilon_{ijk}| J_k$$
$$\mathcal{X} \in so(3) \simeq \mathbb{R}^3: \qquad \mathcal{X}_{ij} = \varepsilon_{ijk} y_k$$

where $|\varepsilon_{ijk}|$ means the absolute value of ε_{ijk}.

Proposition 3. *At $c_2 = 0$ the equations of motion (14) on the sphere S^2 generated by the Hamilton function (13) can be written in the Lax form (15) with the following matrices:*

$$L = \lambda I + \mathcal{J} + a\mathcal{X} \qquad M_{ij} = \frac{2a}{3}|\varepsilon_{ijk}| x_k^{-1}. \tag{18}$$

More explicitly, the first matrix is

$$L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & J_3 & J_2 \\ J_3 & 0 & J_1 \\ J_2 & J_1 & 0 \end{pmatrix} + a \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}$$

and the second matrix is given by

$$M = \frac{2ac_1^{1/2}}{3\sqrt{(y,y)}} \begin{pmatrix} 0 & y_3^{-1} & y_2^{-1} \\ y_3^{-1} & 0 & y_1^{-1} \\ y_2^{-1} & y_1^{-1} & 0 \end{pmatrix}.$$

The spectral invariants of $L(\lambda)$ give rise to both integrals of motion in involution (13)

$$\det L(\lambda) = \lambda^3 - H\lambda + 2K.$$

The proposed Lax matrix $L(\lambda)$ has a trivial dependence on the spectral parameter λ, which is similar to the Lax matrix for the Kowalewski top derived by Perelomov [13]. Therefore, we cannot construct a suitable spectral curve and cannot directly integrate the equations of motion. Recall that, in [13] the Perelomov matrices were embedded into the general Lax matrices with the spectral parameter. It forces us to consider the Lax representation (18) as a first attempt to build an adequate Lax pair. We believe the desired Lax pair explains the peculiar geometry and the origin of integrability of the considered motion on the sphere S^2.

3. Linearization procedure

To conclude this paper we briefly discuss the results obtained in [10] within the modern theory of linearization of the two-dimensional integrable systems [1, 4, 16]. The integration procedure proposed in [10] has an unusual form, which is closely related to the concrete system of equations. This procedure may be related to the work by Chaplygin [7] dealing with the Kirchhoff equations at $c_2 = 0$. On the other hand, the modern theory of linearization allows us to consider different integrable systems such as the Neumann problem, the Henon–Hailes system, the Toda lattice, the Kowalewski top, the Goryachev–Chaplygin top and many others [1, 4, 16].

However, in this common but powerful method it is necessary to rewrite equations of motion at some suitable variables. These variables have to satisfy the special conditions [1, 4, 16]. As an example, to integrate the Toda lattice we have to introduce so-called Flaschka variables. For other systems such variables may be introduced by using the Kowalewski–Painlevé analysis or the algebro-geometric tools. Nevertheless, if we have introduced such variables, the Adler and van Moerbeke methods [1, 4, 16] enable us to integrate a given mechanical system.

The aim of this section is to introduce an analogue of the Flaschka variables for the deformations of the spherical top. At these variables we may directly apply the Adler and van Moerbeke methods to a given integrable system. These results will be presented in future publications.

The three-body Toda lattice is the Hamiltonian system defined as

$$H = \frac{1}{2}\sum_{j=1}^{3} p_j^2 + \mathrm{e}^{q_1-q_2} + \mathrm{e}^{q_2-q_3} + \mathrm{e}^{q_3-q_1}.$$

Here $\{p_j, q_j\}_{j=1}^3$ are pairs of canonical physical variables. According to [1, 4, 16], in the Flaschka variables

$$\begin{aligned} &z_1 = \mathrm{e}^{q_1-q_2} \qquad z_2 = \mathrm{e}^{q_2-q_3} \qquad z_3 = \mathrm{e}^{q_3-q_1} \qquad z_1z_2z_3 = 1\\ &z_4 = -p_1 \qquad z_5 = -p_2 \qquad z_6 = -p_3 \end{aligned}$$

the corresponding equations of motion have the following form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}z_1 &= z_1(z_5 - z_4) \qquad & \frac{\mathrm{d}}{\mathrm{d}t}z_4 &= z_1 - z_3\\ \frac{\mathrm{d}}{\mathrm{d}t}z_2 &= z_2(z_6 - z_5) \qquad & \frac{\mathrm{d}}{\mathrm{d}t}z_5 &= z_2 - z_1\\ \frac{\mathrm{d}}{\mathrm{d}t}z_3 &= z_3(z_4 - z_6) \qquad & \frac{\mathrm{d}}{\mathrm{d}t}z_6 &= z_3 - z_2. \end{aligned} \tag{19}$$

The Toda flow has the following four constants of motion:

$$\begin{aligned} Z_1 &= z_1z_2z_3 = 1\\ Z_2 &= z_4 + z_5 + z_6 = d_1 = 0\\ Z_3 &= \tfrac{1}{2}(z_4^2 + z_5^2 + z_6^2) + z_1 + z_2 + z_3 = a_1\\ Z_4 &= z_4z_5z_6 - z_1z_6 - z_2z_4 - z_3z_5 = b_1. \end{aligned} \tag{20}$$

At $d_1 \neq 0$ the variable $Q = q_1 + q_2 + q_3$ cannot be restored from variables $\{z\}_{j=1}^6$. Really, we have to add to the $\{z_j\}$ variables some other variables with trivial dynamics [1]. Below we introduce the analogue of the Flaschka variables for the integrable deformations of the spherical top.

There is a discrete permutation group acting on the vectors q and p simultaneously:

$$q \longmapsto \mathcal{D}q \qquad p \longmapsto \mathcal{D}p \qquad \mathcal{D} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix} \mathcal{D}^3 = 1. \tag{21}$$

According to this point symmetry, the invariant manifold defined by (20) has a third-order automorphism given by

$$(z_1, z_2, z_3, z_4, z_5, z_6) \longmapsto (z_2, z_3, z_1, z_5, z_6, z_4). \tag{22}$$

This automorphism simplifies the Adler and van Moerbeke analysis [1, 4], which applied to this system, gives linearization to the Toda flow.

Recall that the Kowalewski–Painlevé analysis also enables us to integrate equations of motion for the Goryachev–Chaplygin top [4, 16]. It is another integrable system on $e^*(3)$ at $c_2 = 0$. In this case we have to introduce some seven-dimensional system with the five constants of motion. Then this seven-dimensional system may be reduced to the Toda system [4]. The similar relations between the Toda flow and the integrable system on $e^*(3)$ are discussed in [14]. Now we want to compare the Toda flow with another integrable system on $e^*(3)$.

Let us turn to the deformation of the completely symmetric top (10). In [9], the following transformation of the independent time variable was proposed:

$$t \to u : \qquad \frac{\mathrm{d}}{\mathrm{d}u} = \frac{4}{3}(y, y)\frac{\mathrm{d}}{\mathrm{d}t} \tag{23}$$

because of the 'weak' Kowalewski–Painlevé criterion. In the new time variable the initial Euler–Poisson equations (14) are

$$\frac{\mathrm{d}}{\mathrm{d}u}J = \frac{a^2}{2}y \times y^{-1} \qquad \frac{\mathrm{d}}{\mathrm{d}u}y = -\frac{1}{2}J \times y^{-1}. \tag{24}$$

Namely, these equations were integrated in hyperelliptic quadratures in [10].

If we want to compare the integrable system on the sphere S^2 with the Toda system, note that equations (19) are invariant due to

$$z_j \longmapsto \frac{1}{z_j} \qquad z_{j+3} \longmapsto -z_{j+3} \qquad j = 1, 2, 3. \tag{25}$$

The first equation in (24) has a similar property due to

$$y_j \longmapsto \frac{1}{y_j} \qquad J_j \longmapsto -J_j \qquad j = 1, 2, 3.$$

Using these observations we introduce the analogues of the Flaschka variables

$$\begin{array}{llll} s_1 = y_1^{-2} & s_2 = y_3^{-2} & s_3 = y_2^{-2} & s_1 s_2 s_3 = 1 \\ s_6 = y_1 J_1 & s_4 = y_3 J_3 & s_5 = y_2 J_2 & \end{array}$$

satisfying the following equations:

$$\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}u}s_1 = s_1(s_5 - s_4) & \dfrac{\mathrm{d}}{\mathrm{d}u}s_4 = \dfrac{a^2}{2}(s_1 - s_3) + \dfrac{s_4}{2}(s_5 - s_6) \\ \dfrac{\mathrm{d}}{\mathrm{d}u}s_2 = s_2(s_6 - s_5) & \dfrac{\mathrm{d}}{\mathrm{d}u}s_5 = \dfrac{a^2}{2}(s_2 - s_1) + \dfrac{s_5}{2}(s_6 - s_5) \\ \dfrac{\mathrm{d}}{\mathrm{d}u}s_3 = s_3(s_4 - s_6) & \dfrac{\mathrm{d}}{\mathrm{d}u}s_6 = \dfrac{a^2}{2}(s_3 - s_2) + \dfrac{s_6}{2}(s_4 - s_5). \end{array} \tag{26}$$

The first column of equations is completely coincident with the corresponding Toda equations (19). The second columns differ since the polynomials are at most quadratic. Thus, for these equations we can directly apply the linearization procedure devised by Adler and van Moerbeke [1,4]. Freedom in the definition of the s_j variables may be used to make the Laurent solutions a bit simpler.

The constants of motion for the flow (26) are given by

$$\begin{aligned} S_1 &= s_1 s_2 s_3 = 1 \\ S_2 &= s_4 + s_5 + s_6 = c_2 = 0 \\ S_3 &= (s_4^2 s_2 + s_5^2 s_3 + s_6^2 s_1) - a^2(s_1 s_2 + s_1 s_3 + s_2 s_3) = a_1 \\ S_4 &= s_4 s_5 s_6 + a^2(s_1 s_6 + s_2 s_4 + s_3 s_5) = b_1. \end{aligned} \tag{27}$$

The three constants S_1, S_2 and S_4 coincide with the corresponding Toda constants. The Hamiltonian S_3 is now a cubic polynomial.

As for the Toda lattice, new equations and constants of motion are invariant under permutation (21). Thus, the invariant manifold defined by (27) possesses the third-order automorphism (22), which simplifies the linearization procedure.

It may, of course, be quite difficult to find variables similar to $\{z_j\}$ or $\{s_j\}$ associated with a given mechanical system. These variables have to satisfy some special conditions [1, 16]. For instance, the corresponding equations of motion must only include polynomials which are at most of second order, see (19), (26). However, if we can introduce such variables, the Adler and van Moerbeke methods [1, 4, 16] enable us to integrate a given mechanical system.

Thus, for motion on sphere (13) one has to embed the affine invariant surface defined by (27) into the projective space, whose closure is a principally polarized Abelian surface. It enables one to define the system in linearizing variables. Then we have to prove that the vector field corresponding to S_4 (27) gives the highest flow with respect to the same hyperelliptic curve of genus two. This will complete the linearization of the integrable deformation of the spherical top. Of course, this general machinery leads to the particular results obtained in [10].

4. Conclusion

In the algebro-geometric approach due to Adler and van Moerbeke [1, 4, 16], the algebraic curve may be constructed without any Lax pair representation. For the considered motion on a sphere (10), by substituting the Laurent solutions into the invariants (27) one gets the hyperelliptic curve [10]. Starting with this curve and the linearizing variables [10] the 2×2 Lax pair may be obtained (see [16] for a review).

In this paper we tried to construct the Lax pair in the framework of the group-theoretical approach to an integrable system [5, 13]. Applying the method of finite-band integration to the adequate Lax matrix, we hope to get solutions, which may be simpler than the original formulae [10], as for the Kowalewski top [5].

Further properties of the integrable deformation of the completely symmetric top, such as action-angles variables, Poisson structures of the seven-dimensional system and the separation of variables are under study. The results and the more detailed geometric description will be published elsewhere.

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